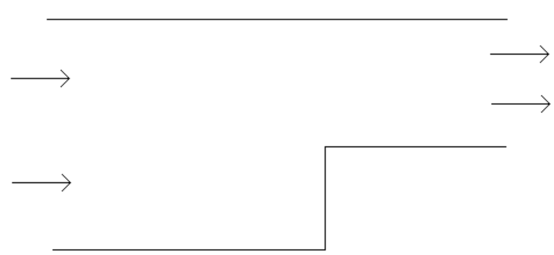
Physics 4931B – Physical Fluid Dynamics
Assignment 2
Due Date: Feb. 16, 2009

Late assignments will be docked marks.

1. Exact solutions to the equations describing a flow can only be obtained in cases where the geometry is very simple. In most cases solutions must be found numerically. The attached pages (taken from *Fluid Mechanics*, by Kundu and Cohen) describe how to solve Laplace's equation numerically in two dimensions (You can also look in the notes for Phys 3926). Using the methods described (or a better method, if you know of one), find the stream function ψ for two-dimensional irrotational flow in a channel with a contraction, as shown in the figure. The flow velocity is uniform and parallel across the input and output sections of the channel, and the flow rate per unit depth is 5 m^2/s. It turns out (although we did not discuss this in class) that the difference in ψ between two streamlines is equal to the flow rate between them, so we can set $\psi = 0$ on the bottom of the channel and $\psi = 5$ on the top. Since the flow is uniform at the inlet and outlet, $u = \frac{\partial \psi}{\partial y}$ must be constant along the left and right ends of the channel, so ψ has to increase linearly from 0 to 5 at each end. This gives you the boundary conditions you need. Use 30 grid points in the y direction and 50 in the x direction, and make the contraction have a height of half the original channel height and a length of 40% of the channel length. Solve Laplace's equation to find ψ. Plot an image of ψ with streamlines superimposed on it (using Matlab's *contour* function, for example). Take the appropriate derivatives of ψ (numerically) to get the x and y components of the flow velocity, and display them in an instructive way.



2. Consider a viscous fluid confined between two infinite horizontal plates at $z = 0$ and $z = d$. There are no external pressure gradients. The lower plate is at rest, and the upper plate rotates about the z axis with an angular frequency Ω. Show that in general a steady solution of the Navier-Stokes equations of the form

$$\overline{u} = u_\theta(r,z)\hat{\theta}$$

is *not* possible. This means that any motion in the azimuthal (θ) direction must generate a secondary flow in which the radial and/or vertical components of the velocity are nonzero.

If the Reynolds number $\mathrm{Re} = \frac{\Omega d^2}{\nu}$ is small, however, such a solution *is* possible! If you are interested (bonus marks!) and enjoy messing around in polar coordinates, try Problem 7.1 in Acheson.

When you stir a cup of tea, the tea leaves all accumulate in the middle of the bottom of the cup. Describe and sketch the flow that must occur in this case, and relate it to the discussion above.

3. Consider the steady, fully developed laminar flow of a viscous fluid flowing in a tube of radius a. This is called circular Poiseuille flow. Using cylindrical coordinates, find the axial flow velocity (which is a function of r, of course) and show that the volume of fluid passing through the pipe per second is proportional to a^4.

4. A fluid fills the half-space $y \geq 0$ bounded by a solid surface at $y = 0$. Sketch the 2D flow near a stagnation point at the origin described by $u = ax$ and $v = -ay$. Show that this flow satisfies the equations for *inviscid* flow.

Unfortunately this flow does not obey the no-slip boundary condition at $y = 0$. To fix this, we introduce a model for the viscous boundary layer in which

$$u = axf'(\eta), \qquad v = -\sqrt{a\nu}\,f(\eta) \quad \text{with } \eta = y\sqrt{a/\nu}.$$

Here $f'(\eta) = \frac{df}{d\eta}$. Show that if

$$f''' + ff'' + 1 - f'^2 = 0,$$

with

$$f(0) = f'(0) = 0, \qquad f'(\infty) = 1,$$

then this form of u and v solve the Navier-Stokes equations with the no-slip boundary condition at $y = 0$, and that they reduce to

$$u = ax \text{ and } v = -ay$$

far from the boundary.

If you can (more bonus marks!), solve the above nonlinear ODE for $f(\eta)$ numerically and plot the result. You will find that f goes to zero smoothly as $\eta \to 0$, and that $f' \approx 1$ for $\eta \geq 3$, so that this model leads to an obvious boundary layer profile for u.

5. A rectangular plate is pulled out of a bath of fluid and held vertically so that the thin layer of liquid can drain off. Assume that the film is initially of uniform thickness h_0. Using the equations for thin-film flow, show that the thickness of the film is given by

$$
\begin{aligned}
h(x,t) &= \left(\frac{\nu x}{gt}\right)^{1/2} && \text{for } 0 < x < \frac{g h_0^2}{\nu} t, \\
&= h_0 && \text{for } x > \frac{g h_0^2}{\nu} t.
\end{aligned}
$$

Here x is measured downwards from the top of the plate. Plot $h(x)$ for a few values of t.