Experiment 4: Damped Oscillations and Resonance in RLC Circuits

Goals:

An RLC circuit is a damped harmonically oscillating system, where the voltage across the capacitor is the oscillating quantity. In the first part of this lab, you will experiment with an *underdamped* RLC circuit and find the decay constant, β, and damped oscillation frequency, ω_1, for the transient, unforced oscillations in the system. In the second part, you will drive the RLC circuit with a sinusoidal voltage and find the resonance frequency, ω_{res}, and amplitude, A_{res}, of the system. The bandwidth and quality factor of the system, $\Delta\omega$ and Q, respectively, will also be found, along with the circuit's response curve.

Procedure:

The first objective is to construct an RLC circuit, which includes a resistor, inductor, and capacitor in series, along with a signal generator:

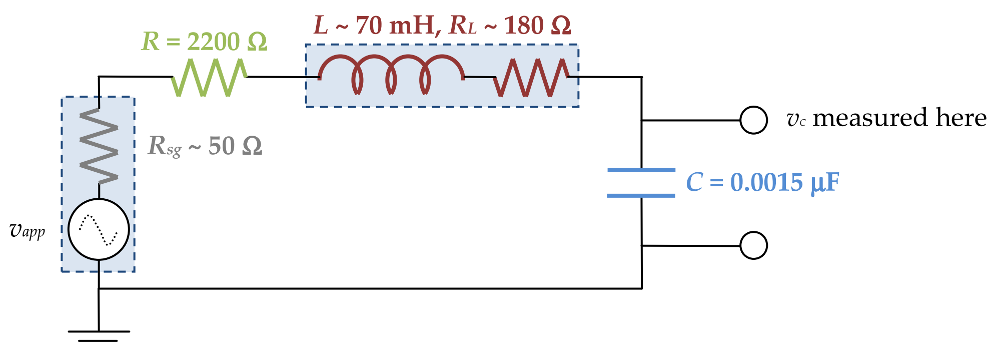

Figure 1: Schematic diagram of RLC system with suggested *R*, *L*, and *C* values

An oscilloscope will be used to measure the applied voltage waveform directly from the signal generator, $v_{app}(t)$, and the voltage across the capacitor, $v_C(t)$. Use components with approximately the specifications given in Figure 1 for best results.[1]

Why is this circuit particularly interesting? You may have seen in class that this circuit behaves like a damped harmonic oscillator, or a pendulum with friction. If one considers Kirchhoff's law for the voltages along the circuit, we have:

$$v_{app} = v_L + v_C + v_R \tag{1}$$

Now, the voltage across an inductor with intrinsic resistance R_L may be written as:

$$v_L = L\frac{di}{dt} + iR_L \tag{2}$$

where i is the current applied to the circuit. Similarly, for a resistor, the voltage may be written as:

$$v_R = iR \tag{3}$$

[1] If you're interested in experimenting with the effect of varying *R* or *C*, there are variable resistors and capacitors available for use.

To get v_L and v_R in terms of v_C, it may also be recalled that:

$$i = C\frac{dv_C}{dt} \tag{4}$$

Substituting all of this into the Kirchhoff's law for this circuit, and noting the non-zero impedance of the signal generator, R_{sg}, gives:

$$\begin{aligned} v_{app} &= v_L + v_C + v_R + iR_{sg} \\ &= \left(LC\frac{d^2 v_C}{dt^2} + R_L C\frac{dv_C}{dt}\right) + v_C + \left(RC\frac{dv_C}{dt}\right) + \left(R_{sg} C\frac{dv_C}{dt}\right) \end{aligned}$$

One can read R_{sg} given on the signal generator outputs. It is assumed that the capacitance is constant with time. This is a second order linear differential equation:

$$\frac{d^2 v_C}{dt^2} + \frac{R_t}{L}\frac{dv_C}{dt} + \frac{1}{LC}v_C = \frac{1}{LC}v_{app} \tag{5}$$

The resistances are gathered as $R_t = R + R_L + R_{sg}$. *Note this for later calculations!* This may be compared to the second order differential equation describing the oscillations of a harmonic oscillator or pendulum:

$$\frac{d^2 x}{dt^2} + 2\beta\frac{dx}{dt} + \omega_0{}^2 x = \frac{F(t)}{m} \tag{6}$$

In this equation, β, the damping or decay constant, describes the damping strength, or how apparent friction is in the system. A large β means that oscillations are stopped rapidly, while $\beta \to 0$ gives an infinitely-oscillating system:[2]

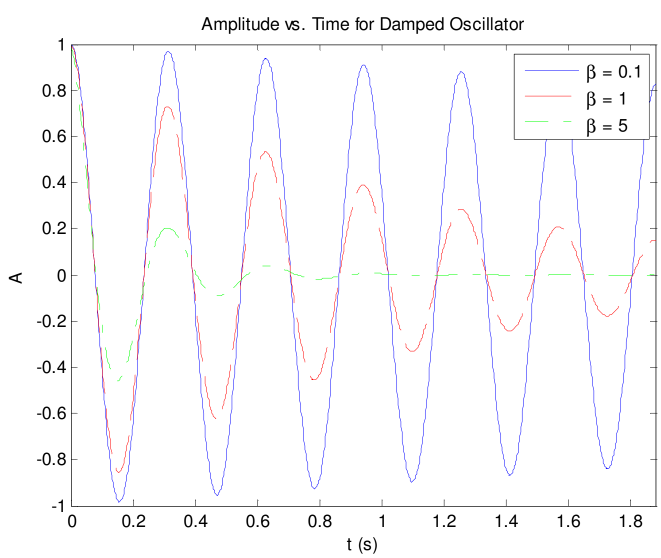

Figure 2: Oscillations in a damped harmonic oscillator for various damping constants

[2] If β is large enough, particularly if $\beta^2 > 1/LC$, the system doesn't even oscillate any more, it becomes *overdamped*. This can be useful for situations where you don't want oscillations in a spring, like for springs on a vehicle's suspension system.

The other parameter, ω_0, is the natural frequency of the system; that is, if the damping is reduced to almost zero, the system would oscillate with frequency ω_0. In general, even with damping, the oscillation frequency is almost equal to ω_0. If we compare the RLC and harmonic oscillator equations, it can be seen that:

$$2\beta = \frac{R_t}{L} \qquad\qquad \omega_0 = \frac{1}{\sqrt{LC}}$$

Thus, the damping of the signal is proportional to the resistance in the circuit, and the natural frequency of oscillation is inversely proportional to L and C.

If we solve the RLC differential equation for no applied voltage, or $v_{app} = 0$, the voltage measured across the capacitor as a function of time is given by:

$$v_{C,T}(t) = \exp\left(-\frac{R_t}{2L}t\right)\cdot\left[A_1 \sin\left(\sqrt{\frac{1}{LC}-\frac{1}{4}\left(\frac{R_t}{L}\right)^2}\cdot t\right) + A_2 \cos\left(\sqrt{\frac{1}{LC}-\frac{1}{4}\left(\frac{R_t}{L}\right)^2}\cdot t\right)\right] \tag{7}$$

Hence the voltage across the capacitor is an exponentially-decaying sinusoid. We can measure this and compare it to theoretical values. Make sure everything is connected correctly in your RLC circuit, and then set the signal generator to create a square wave signal with voltage ~ 5 V, and frequency ~ 300 Hz. You will want a waveform that resembles the following:

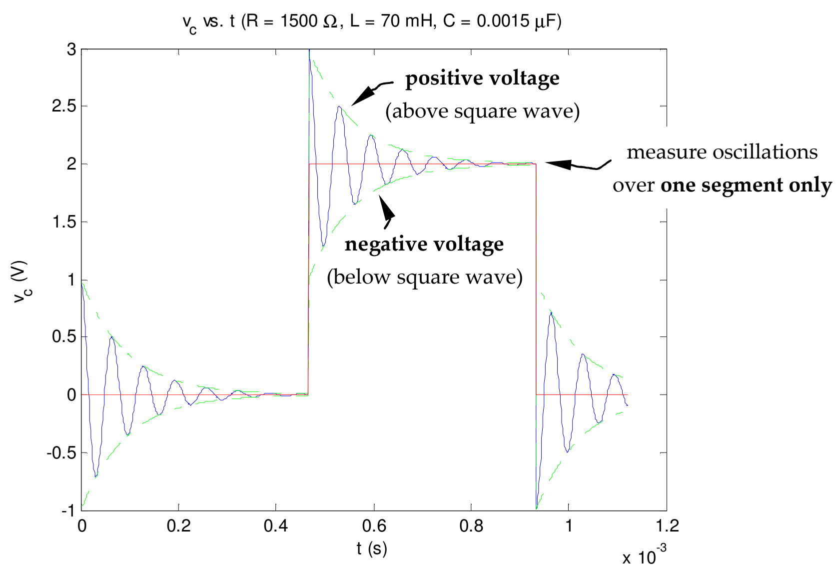

Figure 3: Transient oscillations in voltage across the capacitor (blue), along with the signal generator's voltage (red); note that the exponential decay envelope is highlighted in green

The red step waveform is the output from the signal generator, while the blue waveform is the output from the RLC circuit. From here, you can see the damped oscillations and decay of the system. Measure the frequency of oscillations using the cursor (since the oscillation is decaying, it's difficult to use the frequency measuring feature of the oscilloscope). For every peak and trough, use the cursor to also measure the amplitude – we can use this to estimate the decay constant. *Note that the amplitude should be measured with respect to the signal generator's driving voltage, and only for one segment of the square wave!* Voltages above the square wave are positive, while voltages below should be negative. With these observations, you may plot the natural logarithm of the absolute value of the voltages you measured versus time to get a linear graph with slope related to $\beta = R_t/2L$ (see Eq. (7)). Plot at least ten points. By looking at the time between a trough and a peak (or multiple troughs and peaks), you may also determine the period of damped oscillation, giving the frequency of damped oscillation. Compare your measured damped oscillation frequency, ω_1, and decay constant, β, to the theoretical values derived from Eq. (7):

$$\omega_1 = \sqrt{\frac{1}{LC} - \frac{1}{4}\left(\frac{R_t}{L}\right)^2} \tag{8}$$

$$\beta = \frac{R_t}{2L} \tag{9}$$

Now that the transient oscillatory nature of the RLC circuit has been examined, we can now drive the circuit with sinusoidal applied voltages of varying frequencies. We previously solved the RLC differential equation with no applied voltage, $v_{app} = 0$, to find the transient behaviour of the system. In this case, we try a sinusoidal v_{app} with angular frequency ω_{app}:

$$v_{app}(t) = v_0 \sin(\omega_{app} t) \tag{10}$$

The solution, giving the voltage across the capacitor, is also a sine function with the same frequency as the applied voltage, but there's a phase shift δ and a frequency-dependent amplitude, A:[3]

$$v_{C,F}(t) = A \sin(\omega_{app} t + \delta) \tag{11}$$

$$A = \frac{v_0}{C} \frac{1}{\sqrt{R_t^2 \omega_{app}^2 + L^2(\omega_{app}^2 - 1/LC)^2}} \tag{12}$$

$$\delta = \tan^{-1}\left[\frac{R_t \omega_{app}}{L(\omega_{app}^2 - 1/LC)}\right] \tag{13}$$

One may find the resonance frequency for the response waveform $v_{C,F}$ by differentiating A with respect to ω_{app} and setting the result equal to zero:

$$\left.\frac{\partial A}{\partial \omega_{app}}\right|_{\omega_{app} = \omega_{res}} = 0 \tag{14}$$

Once the resonance frequency, ω_{res}, is found it may be substituted into A to find the resonance amplitude, or the maximum voltage across the capacitor that will be observed for the RLC circuit. We can compare theory to observations by setting the wave generator to produce a sinusoidal waveform of varying frequencies and comparing the wave generator's peak-to-peak driving voltage, v_{app}, to the circuit's peak-to-peak response voltage, $v_{C,F}$. Some sample results are given in Figure 4, on the next page.

[3] See *Classical Dynamics of Particles and Systems*, 5th Edition, by Marion and Thornton for a detailed solution of Eq. (5) giving Eq. (11), (12), and (13)

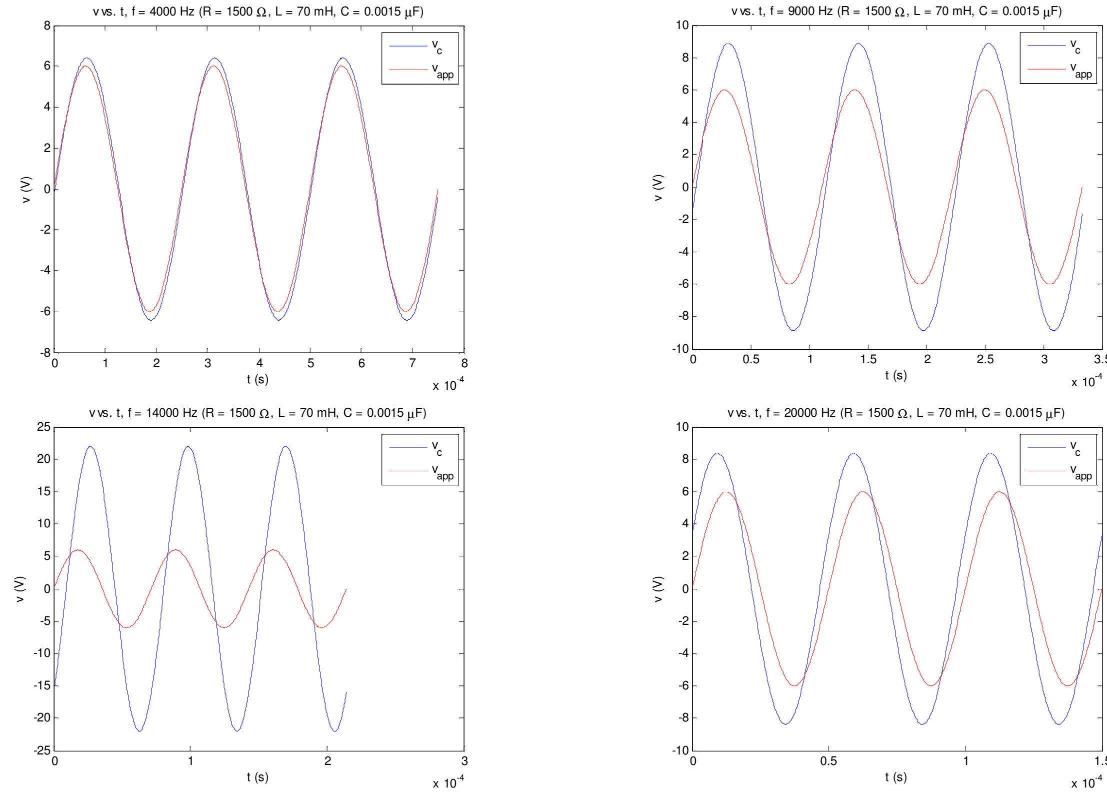

Figure 4: Oscilloscope readings for applied voltage signals of various frequencies; note that the forcing and response voltages have almost equal magnitudes for small frequencies (< 5000 Hz), but as they approach the resonance frequency, they become much larger than the forcing voltages!

For the circuit with the *R*, *L*, and *C* values given in Figure 1, resonance will occur at a frequency around 15000 Hz, and the resonance amplitude should be about three-times the input amplitude. Vary the frequency from about 100 Hz to 25000 Hz in steps that will allow you to plot a smooth graph of response voltage versus applied frequency. This is called the circuit's *response curve*, and indicates how selective a circuit is when filtering frequencies. For low and high frequencies, say below 5000 Hz and above 17500 Hz, you may not need as many points, but to get the best results, you'll need to collect a *large* number of points around the resonance frequency. For reference, 25 data points from 20 Hz to 23000 Hz, with most of the points collected around 15000 Hz, will definitely produce a nice, smooth response curve. Next, find the frequency at which the maximum amplitude (measure this too) is observed. This can be compared to the theoretical resonance frequency and amplitude that you may calculate from the given expression for *A*, above.

Another item of interest is the quality factor, or *Q*-factor, defined as the following:

$$Q \equiv \frac{\omega_{res}}{2\beta} = \omega_{res}\frac{L}{R_t} \tag{15}$$

The *Q*-factor is a measure of how strongly a system is damped. A large *Q*-factor means that the RLC circuit selects a small band of frequencies to amplify by a large ratio, making it a good band-pass filter. For small resistances:

$$Q \approx \frac{\omega_0}{\Delta\omega} \sim \frac{\omega_{res}}{\Delta\omega} \tag{16}$$

where $\Delta\omega$ is the bandwidth of the circuit, the difference between the two frequencies that produce a capacitor voltage that is $1/\sqrt{2}$ the maximum oscillatory voltage, A_{res}. The Q-factor may be calculated from observations by determining the bandwidth $\Delta\omega$ and using the observed resonance frequency ω_{res}, for Eq. (16). This may be compared to the theoretical value, given in Eq. (15).

Conclusion:

For your writeup, make sure that you have the following observations, and compare them, quantitatively, to theoretical equivalents given in this literature:

- decay constant, β (plot $\ln(v_c)$ vs. t for the first part of the experiment, and include this graph)
- damped oscillatory frequency, ω_1

- resonance amplitude, A_{res}
- resonance frequency, ω_{res}
- bandwidth, $\Delta\omega$
- quality factor, Q (since Eq. (16) is an approximation, there will be *some* error, here)

If there are discrepancies between observations and expectations, comment on sources of error, and possible improvements to the experimental method. Estimate the error in each observed quantity and justify your estimate.

You should also have a graph of the response curve for the second part of the experiment, giving the peak-to-peak response voltage versus frequency when the circuit is driven by a sinusoidal signal from the signal generator:

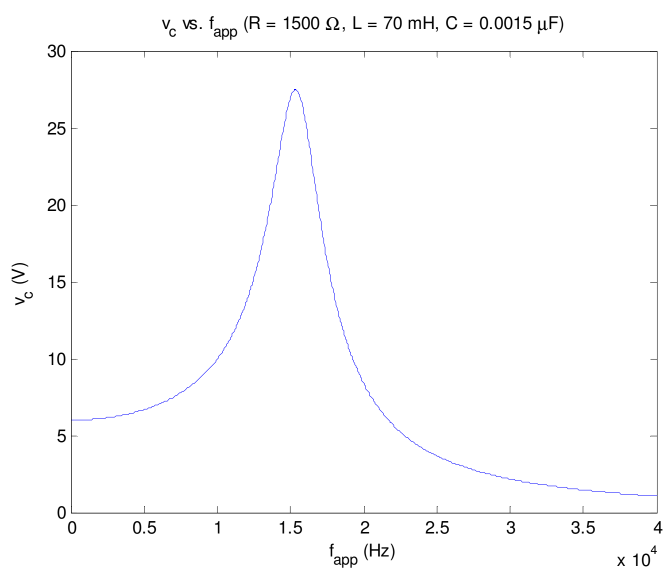

Figure 5: Theoretical response curve for an RLC circuit

Plot the theoretical curve (see Eq. (12)) on the same graph as the experimental results, using a solid line for the theoretical expectation and points for the experimental observations. Comment on the agreement of theory and experiment.

(No formal writeup is required, just the observations, calculations, graphs, and commentary listed above)